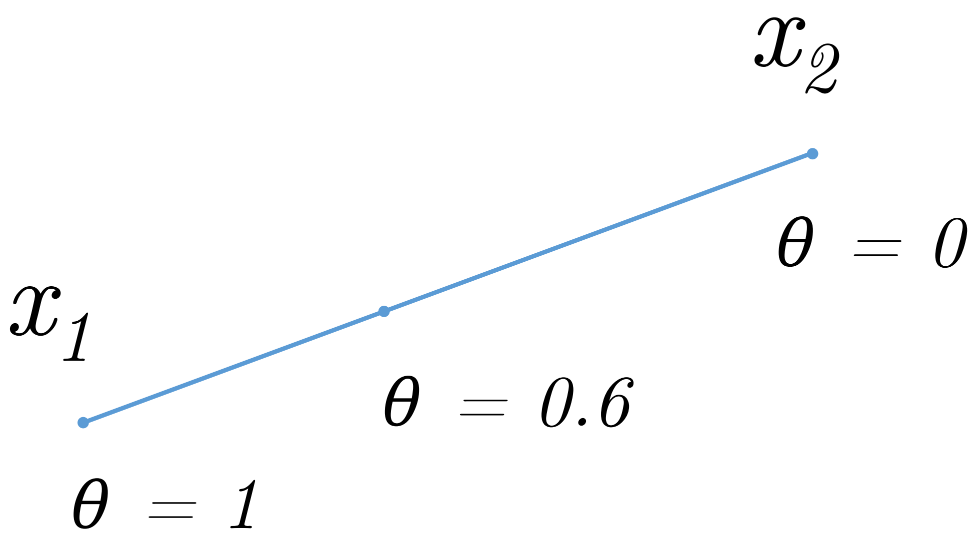
Convex set

Line segment

Suppose x_1, x_2 are two points in $\mathbb{R}^n$. Then the line segment between them is defined as follows:

$$x = \theta x_1 + (1 - \theta)x_2, \ \theta \in [0,1]$$



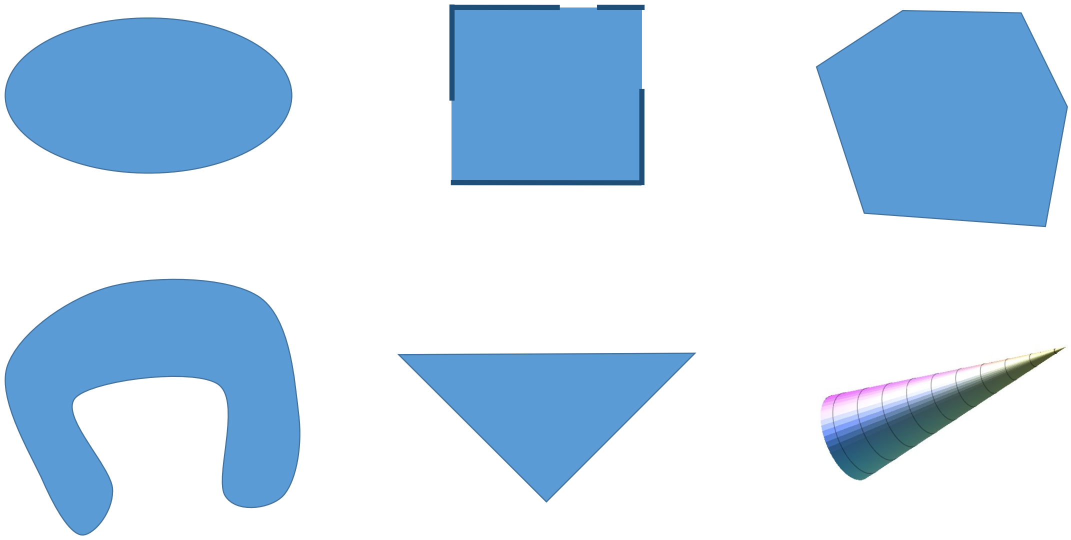
Convex set

The set S is called **convex** if for any x_1, x_2 from S the line segment between them also lies in S, i.e.

$$\forall \theta \in [0,1], \ \forall x_1, x_2 \in S: \\ \theta x_1 + (1-\theta)x_2 \in S$$

Examples:

- Any affine set
- Ray
- Line segment



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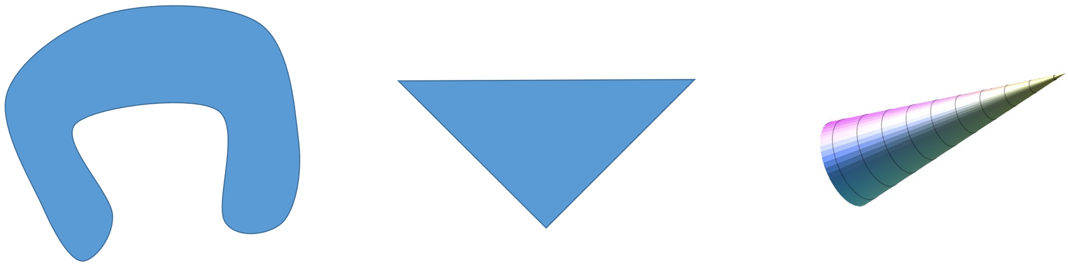

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Related definitions

Convex combination

Let $x_1, x_2, \ldots, x_k \in S$, then the point $\theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k$ is called the convex combination of points $x_1, x_2, \ldots, x_k$ if $\sum\limits_{i=1}^k \theta_i = 1, \; \theta_i \geq 0$

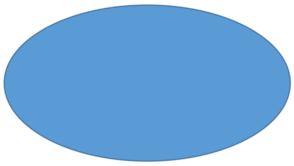
Convex hull

The set of all convex combinations of points from S is called the convex hull of the set S.

$$\mathbf{conv}(S) = \left\{ \sum_{i=1}^k \theta_i x_i \mid x_i \in S, \sum_{i=1}^k \theta_i = 1, \; \theta_i \geq 0 \right\}$$

- The set $\mathbf{conv}(S)$ is the smallest convex set containing S.
- The set S is convex if and only if $S = \mathbf{conv}(S)$.

Examples:



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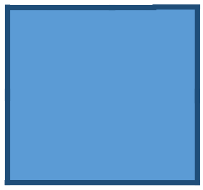

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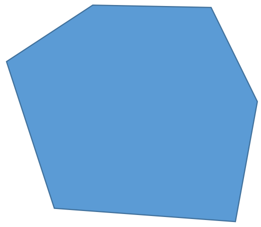

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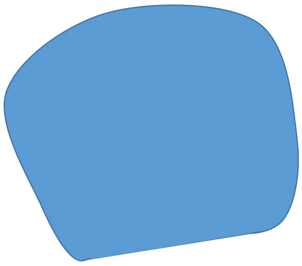

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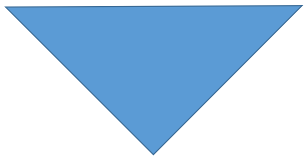

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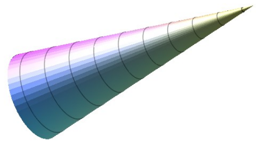

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Finding convexity

In practice it is very important to understand whether a specific set is convex or not. Two approaches are used for this depending on the context.

- By definition.
- Show that S is derived from simple convex sets using operations that preserve convexity.

By definition

$$x_1, x_2 \in S, \; 0 \le \theta \le 1 \quad \rightarrow \quad \theta x_1 + (1-\theta)x_2 \in S$$

Preserving convexity

The linear combination of convex sets is convex

Let there be 2 convex sets S_x, S_y, let the set $S = \{s \mid s = c_1 x + c_2 y, \; x \in S_x, \; y \in S_y, \; c_1, c_2 \in \mathbb{R}\}$

Take two points from S: $s_1 = c_1 x_1 + c_2 y_1, s_2 = c_1 x_2 + c_2 y_2$ and prove that the segment between them $${theta s_1 + (1 - \theta)s_2, \theta \in [0,1]$$ also belongs to $$S$$

$$\theta s_1 + (1-\theta)s_2$$

$$\theta(c_1 x_1 + c_2 y_1) + (1-\theta)(c_1 x_2 + c_2 y_2)$$

$$c_1(\theta x_1 + (1-\theta)x_2) + c_2(\theta y_1 + (1-\theta)y_2)$$

$$c_1 x + c_2 y \in S$$

The intersection of any (!) number of convex sets is convex

If the desired intersection is empty or contains one point, the property is proved by definition. Otherwise, take 2 points and a segment between them. These points must lie in all intersecting sets, and since they are all convex, the segment between them lies in all sets and, therefore, in their intersection.

The image of the convex set under affine mapping is convex

$$S \subseteq \mathbb{R}^n \text{ convex} \quad \to \quad f(S) = \{f(x) \mid x \in S\} \text{ convex} \qquad (f(x) = \mathbf{A}x + \mathbf{b})$$

Examples of affine functions: extension, projection, transposition, set of solutions of linear matrix inequality $\{x \mid x_1 A_1 + \ldots + x_m A_m \preceq B\}$ Here $A_i, B \in \mathbf{S}^p$ are symmetric matrices $p \times p$.

Note also that the prototype of the convex set under affine mapping is also convex.

$$S \subseteq \mathbb{R}^m \text{ convex} \;\to\; f^{-1}(S) = \{x \in \mathbb{R}^n \mid f(x) \in S\} \text{ convex} \;\; (f(x) = \mathbf{A}x + \mathbf{b})$$

Example 1

Prove, that ball in $\mathbb{R}^n$ (i.e. the following set $\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_c\| \leq r\}$) - is convex.

Example 2

Which of the sets are convex: 1. Stripe, $\{x \in \mathbb{R}^n \mid \alpha \leq a^\top x \leq \beta\}$ 1. Rectangle, $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = \overline{1, n}\}$ 1. Kleen, $\{x \in \mathbb{R}^n \mid a_1^\top x \leq b_1, a_2^\top x \leq b_2\}$ 1. A set of points closer to a given point than a given set that does not contain a point, $\{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leq \|x - y\|_2, \forall y \in S \subseteq \mathbb{R}^n\}$ 1. A set of points, which are closer to one set than another, $\{x \in \mathbb{R}^n \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T), S, T \subseteq \mathbb{R}^n\}$ 1. A set of points, $\{x \in \mathbb{R}^n \mid x + X \subseteq S\}$, where $S \subseteq \mathbb{R}^n$ is convex and $X \subseteq \mathbb{R}^n$ is arbitrary. 1. A set of points whose distance to a given point does not exceed a certain part of the distance to another given point is $\{x \in \mathbb{R}^n \mid \|x - a\|_2 \leq \theta \|xb\|_2, a, b \in \mathbb{R}^n, 0 \leq 1\}$

Example 3

Let $x \in \mathbb{R}$ is a random variable with a given probability distribution of $\mathbb{P}(x = a_i) = p_i$, where $i = 1, \ldots, n$, and $a_1 < \ldots < a_n$. It is said that the probability vector of outcomes of $p \in \mathbb{R}^n$ belongs to the probabilistic simplex, i.e.
$P = \{p \mid \mathbf{1}^T p = 1, p \succeq 0\} = \{p \mid p_1 + \ldots + p_n = 1, p_i \ge 0\}$. Determine if the following sets of p are convex: 1. $\alpha < \mathbb{E} f(x) < \beta$, where $\mathbb{E} f(x)$ stands for expected value of $f(x) : \mathbb{R} \to \mathbb{R}$, i.e. $\mathbb{E} f(x) = \sum\limits_{i=1}^n p_i f(a_i)$ 1. $\mathbb{E} x^2 \le \alpha$ 1. $\mathbb{V} x \le \alpha$

Convex function

Convex function

The function $f(x)$, **which is defined on the convex set** $S \subseteq \mathbb{R}^n$, is called **convex** S, if:

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

for any $x_1, x_2 \in S$ and $0 \le \lambda \le 1$.
If above inequality holds as strict inequality $x_1 \neq x_2$ and $0 < \lambda < 1$, then function is called strictly convex S